B.Sc.(Math) - part II
paper - III
Topic :- uniquenes of identity element and inverse element in group

Theorem ① To prove that the identity element of a group is unique

proof :- Let (G, o) be a group with the identity element $e \in G$.
We have to prove that identity element is unique.
~~Proof~~ :- If possible let $e' \in G$ be another identity element of the group.
As e is the identity element by def $aoe = eoa = a$ for all $a \in G$
As $e' \in G$ putting e' for e in the identity we get

$e'oe = eoe' = e'$ — (1)

By assumption e' is also an identity element of G
$\therefore$ By def $aoe' = e'oa = a$
As $e \in G$, $eoe' = e'oe = e$ — (2)
By (1) and (2) we get $e = e'$
Therefore e is unique identity element.

Theorem (2) - Uniqueness of inverse element

Theorem :- The inverse of an element in a group is unique.

proof :- Let G be a group. Let α be any element of G and let α^{-1} be its inverse.

We have to prove that α^{-1} is unique. If not suppose α' is a another inverse of α.

Since α^{-1} is the inverse of α therefor

$$\alpha\alpha^{-1} = \alpha^{-1}\alpha = e \quad \text{--- (1)}$$

Similarly, since α' is the inver of α therefore

$$\alpha\alpha' = \alpha'\alpha = e \quad \text{--- (2)}$$

Where e is identity element of G.

Multiplying (1) by α' on the left we get

$$\alpha'(\alpha\alpha^{-1}) = \alpha' e = \alpha' \quad \text{--- (3)}$$

Multiplying (2) by α^{-1} on the right we get

$$(\alpha'\alpha)\alpha^{-1} = e\alpha^{-1} = \alpha' \quad \text{--- (4)}$$

But by associative law

$$\alpha'(\alpha\alpha^{-1}) = (\alpha'\alpha)\alpha^{-1}$$

Therefore we have from (3) and (4)

$$\alpha' = \alpha^{-1}$$

Hence the inverse of an elemen in a group is unique.

Theorem ③ To prove that in a Group (G, o), $(aob)^{-1} = b^{-1}oa^{-1}$

proof :- we have to prove that the inverse element of aob is $b^{-1}oa^{-1}$ where $a, b \in G$

By def of inverse, the inverse element of x is y if $xoy = yox = e$, the identity element

So we have to prove

$$(aob)o(b^{-1}oa^{-1}) = (b^{-1}oa^{-1})o(aob) = e$$

Now $(aob)o(b^{-1}oa^{-1}) = ao(bob^{-1})oa^{-1}$ for the associative law hold for a group

$= aoeoa^{-1}$ $(\because bob^{-1} = e)$

$= (aoe)oa^{-1} = aoa^{-1} = e$

$(b^{-1}oa^{-1})o(aob) = b^{-1}o(a^{-1}oa)ob$

$= b^{-1}oeob = b^{-1}o(eob)$

$= b^{-1}ob = e$

Thus we get $(aob)o(b^{-1}oa^{-1})$

$= (b^{-1}oa^{-1})(aob) = e$

So $b^{-1}oa^{-1}$ is the inverse element of aob

ie $(aob)^{-1} = b^{-1}oa^{-1}$